

NON-COINCIDENCE OF QUENCHED AND ANNEALED CONNECTIVE CONSTANTS ON THE SUPERCRITICAL PLANAR PERCOLATION CLUSTER

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ABSTRACT. In this paper, we study the abundance of self-avoiding paths of a given length on a supercritical percolation cluster on \mathbb{Z}^d . More precisely, we count Z_N the number of self-avoiding paths of length N on the infinite cluster, starting from the origin (that we condition to be in the cluster). We are interested in estimating the upper growth rate of Z_N , $\limsup_{N \rightarrow \infty} Z_N^{1/N}$, that we call it connective constant of the dilute lattice. After proving that this connective constant is a.s. non-random, we focus on the two-dimensional case and show that for every percolation parameter $p \in (1/2, 1)$, almost surely, Z_N grows exponentially slower than its expected value. In other word we prove that $\limsup_{N \rightarrow \infty} (Z_N)^{1/N} < \lim_{N \rightarrow \infty} \mathbb{E}[Z_N]^{1/N}$ where expectation is taken with respect to the percolation process. This result can be considered as a first mathematical attempt to understand the influence of disorder for self-avoiding walk on a (quenched) dilute lattice. Our method, which combines change of measure and coarse graining arguments, does not rely on specifics of percolation on \mathbb{Z}^2 , so that our result can be extended to a large family of two dimensional models including general self-avoiding walk in random environment.

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1. MODEL AND RESULT

1.1. Introduction. We are interested in percolation on the \mathbb{Z}^d grid ($d \geq 2$) with its usual lattice structure, E_d denotes the set of edges (they link nearest neighbors). We delete each edge with probability $1 - p$ ($p > 1/2$) and investigate the connectivity properties of the resulting network (or rather of its unique infinite connected component). More precisely we want to study the asymptotic growth of the number of *self-avoiding paths* of length N starting from a given (typical) point on the dilute lattice. A self-avoiding path is a lattice path that does not visit twice the same vertex.

Comparing the number of self-avoiding path with its expected value gives some heuristic information concerning the influence of a quenched edge dilution on the trajectorial behavior of the self-avoiding walk.

1.2. The self-avoiding walk. Let us first recall some fact about self-avoiding walk on a regular lattice (we focus on \mathbb{Z}^d for the sake of simplicity). Set

$$\mathcal{S}_N := \{ \text{self-avoiding paths } (S_n)_{n \in [0, N]} \text{ on } \mathbb{Z}^d \text{ of length } N \text{ starting } 0 \} \quad (1.1)$$

and $s_N := |\mathcal{S}_N|$. As s_N is a submultiplicative function, the limit

$$\lim_{n \rightarrow \infty} (s_N)^{1/N} := \mu_d \quad (1.2)$$

exists.

The constant μ_d is called the connective constant of the network. It is not expected to take any remarkable value as far as \mathbb{Z}^d is concerned (on the two dimensional honeycomb lattice on the contrary it has been conjectured for a long time and has been recently proved that $\mu = \sqrt{2 + \sqrt{2}}$, see [11]).

The self-avoiding walk of length N is a stochastic process whose law is given by the uniform probability measure on \mathcal{S}_N . It has been introduced as a model for polymer by Flory [12]. Theoretical physicists have then been interested in describing typical behavior of the walk for large N , in understanding whether it differs from the one of the simple random walk and why. Their answer to this concern depends on the dimension and is the following:

- (i) When $d > 4$, the self-avoiding constraint is a local one. Indeed, around a typical point of a simple random walks trajectory, the past and the future intersect finitely many times, at a finite distance. For this reason, the self-avoiding walk in dimension larger than 4 scales like Brownian Motion. The case $d = 4$ which corresponds to the critical dimension, should be a bit similar, but with logarithmic corrections.
- (ii) When $d < 4$, the self-avoiding constraint acts also on large scale, and modifies the macroscopic structure of the walk. In particular it forces the walk to go further and the end to end distance $|S_N|$ is believed to scale like N^ν where $\nu = 3/4$ for $d = 2$ and $\nu \simeq 0.59$ for $d = 3$.

On the mathematical side, the picture is much less complete. Above the critical dimension, when $d > 4$, the use of the lace expansion by Brydges and Spencer [4] allowed to make rigorous the physicists prediction, but when $d < 4$, very few things are known rigorously (for a complete introduction to the subject and a list of the conjecture see the first chapter of [23], or [25] for a more recent survey). Note that recently Duminil-Copin and Hammond proved that the self-avoiding walk is non-ballistic in every dimension [10].

1.3. Percolation on \mathbb{Z}^d . Let ω be the edge dilution (or percolation) process defined on the set of the edges of \mathbb{Z}^d as follows

- $(\omega(e))_{e \in E_d}$ is a field of IID $\{0, 1\}$ Bernoulli variable with law denoted by \mathbb{P}_p satisfies $\mathbb{P}_p(\omega(e) = 1) = p$.
- Every edge e such that $\omega(e) = 1$ is declared open or present whereas the others are deleted (or closed).

A set of edges is declared open if all the edges in it are open and a self-avoiding path S is declared open if all the edges in the path are (we will constantly use the informal notation $e \in S$).

The nature of the new lattice obtained after deleting edges crucially depends on the value of p . There exists a constant $p_c(d)$ called percolation threshold such that the dilute lattice contains a unique infinite connected component if $p \in (p_c, 1]$ (in addition to countably many finite connected component), and none if $p < p_c$. It is also known that $p_c(2) = 1/2$ (see e.g. [16] for a complete introduction to percolation).

1.4. Quenched connective constant for the percolation cluster. In what follows we consider exclusively the supercritical regime $p > p_c$. We call \mathcal{C} the supercritical percolation cluster (the unique infinite connected component).

Set Z_N to be the number of open self-avoiding paths of length N starting from the origin:

$$Z_N := \sum_{S \in \mathcal{S}_N} \mathbf{1}_{\{S \text{ is open}\}}. \quad (1.3)$$

Similarly one can define $Z_N(x)$ considering paths that starts from x instead of paths starting from the origin. One has trivially

$$Z_N(x) = 0 \text{ eventually for all } N \Leftrightarrow x \notin \mathcal{C}. \quad (1.4)$$

Our aim is to study the asymptotic behavior of $Z_N(x)$ when x belongs to \mathcal{C} . One can easily compute its expectation: for x in \mathbb{Z}^d

$$\mathbb{E}_p[Z_N(x)] = \mathbb{E}_p[Z_N] = \sum_{S \in \mathcal{S}_N} \mathbb{P}_p[S \text{ is open}] = p^N s_N, \quad (1.5)$$

and this gives an upper bound on the possible growth rate of $Z_N(x)$ (we cannot compute exactly the expectation conditioning to $x \in \mathcal{C}$, but the reader can check that it has the same order of magnitude).

Now we define an equivalent of the connective constant μ_d for the infinite percolation cluster \mathcal{C} . The following result is valid in any dimension.

Proposition 1.1. *For every $x \in \mathcal{C}$, the limit*

$$\limsup_{N \rightarrow \infty} (Z_N(x))^{\frac{1}{N}}, \quad (1.6)$$

does not depend on x and is non-random. We call this limit the quenched connective constant of the dilute lattice and denote it by $\mu_d(p)$. It satisfies the inequality

$$\mu_d(p) \leq p \mu_d(1). \quad (1.7)$$

We call

$$\mathbb{E}_p[Z_N]^{\frac{1}{N}} = p \mu_d(1). \quad (1.8)$$

the annealed connective constant.

Moreover the ratio between quenched and annealed connective constant

$$\mu_d(p)/p \mu_d(1)$$

is a non-increasing function of p on $(p_c, 1]$.

Remark 1.2. We believe that

$$\lim_{N \rightarrow \infty} (Z_N(x))^{\frac{1}{N}}, \quad (1.9)$$

exists but the best we can do here is to state this as a conjecture.

We are interested in knowing whether or not the inequality (1.7) is sharp. The reason for this is that at a heuristic level, the ration $Z_N/\mathbb{E}[Z_N]$ conveys some information on the trajectorial behavior of the self-avoiding walk on the dilute lattice. The self-avoiding walk of length N on the dilute lattice is the stochastic process whose law is given by the uniform probability measure on the random set

$$\mathcal{S}_N(\omega) := \{S \in \mathcal{S}_N \mid S \text{ is open for } \omega\}. \quad (1.10)$$

Note that this definition makes sense for all N only if $0 \in \mathcal{C}$.

In analogy with what happens for directed polymer in a random environment we believe that:

- (i) If $Z_N/\mathbb{E}[Z_N]$ is typically of order 1 then, the self-avoiding walk on the dilute lattice has a behavior similar to the one on the full lattice.
- (ii) If $Z_N/\mathbb{E}[Z_N]$ decays exponentially fast, then disorder changes the behavior of the trajectories. it induces localization of trajectories (they concentrate in regions where ω is more favorable), and possibly stretches the trajectory, making end-to-end distance $|S_N|$ larger.

Using some of the techniques that have been used for directed polymer could bring these statements at least on a more rigorous ground (see [8] for an analogy with case (i) and [5] for an analogy with (ii) for the part concerning localization). Of course, saying something rigorous about the end-to-end distance for the disordered model is quite hopeless as it is already a difficult open question for the homogeneous model.

1.5. Main result. The main result of this paper is that this result is that the quenched connective constant is strictly smaller than the annealed one for the model on \mathbb{Z}^2 , indicating localization of the trajectories.

Theorem 1.3. *For every $p \in (p_c(2), 1)$*

$$\mu_2(p) < p\mu_2(1), \quad (1.11)$$

meaning that $Z_N/\mathbb{E}_p[Z_N]$ tends to zero exponentially fast for any $p < 1$. Moreover the function

$$p \mapsto \frac{\mu_2(p)}{p\mu_2(1)},$$

is (strictly) increasing on $(1/2, 1]$.

Although the proof allows to extract an explicit upper-bound for $\mu_2(p) - p\mu_2(1)$, which gets exponentially small when p is close to one, we believe it to be very far from optimal when edge dilution is small. Indeed if $|S_N|$ scales like N^ν , with $\nu < 1$, the argument of the proof in [21, Section 3] give at a heuristic level that $p\mu_2(1) - \mu_2(p)$ is at least of order $(1 - p)^{\frac{1}{2(1-\nu)}}$ (which is to be compared with the bound in (3.16)).

1.6. Comparison with prediction in the physics literature. Although physics literature concerning Self-avoiding walk on directed lattice is quite rich (for the first paper on the subject see [6], it is difficult to extract solid conjecture on the value of $\mu_d(p)$ from the variety of contributions.

The first reason is that in most of the studies, the focus is on trajectorial behavior, and only a marginal attention is given to the partition function Z_N (to that respect, [1] is a noticeable exception with explicit focus on $\mu_d(p)$).

The second reason, is that while it is often not very clearly stated, it seems that most of the paper from the eighties are focused *annealed* model of self-avoiding walk on a percolation cluster, which seems mathematically trivial (see for instance the sequence of equation to compute the mean square of the end-to-end distance in [17]). In fact the last paragraph of the Introduction in [9] explicitly states that some of the earlier studies are about the annealed model.

Because of this last remark, we do not feel that our result contradicts the many papers (see e.g. [20, 17, 18]) which predict that edge-dilution does not change the walks behavior. The few numerical studies available concerning the connective constant are not very informative either: both [7, Table 1] and the graph [1, Figure 3] gives value for $\mu_2(p)$ that violates the annealed bound ($\mu_d(p) \leq p\mu_d(1)$).

In [9], the authors clearly state that they study the quenched problem, and make a number of prediction, partially based on a renormalization group study performed on hierarchical lattices:

- When $d = 2, 3$, there is no phase transition, and an arbitrary small dilution changes the properties of the self-avoiding walk.
- When $d > 4$, a small edge-dilution does not change the trajectorial property, and there is a phase transition from a weak disorder phase to a strong disorder one when p varies.

Further more they give an explicit formula linking the typical fluctuations of $\log Z_N$ around its mean with the end-to-end exponent ν in the strong disorder phase.

To our understanding, our result partially confirms the prediction of [9] in dimension 2. For the rest of those predictions concerning higher dimension, we would translate them in terms of quenched connective constant as follows:

- $\mu_d(p) < p\mu_d(1)$ for all $p < 1$ for $d = 3$.
- When $d > 4$, there exists $p_c^{(2)}(d) \in (p_c(d), 1)$ such that $\mu_d(p) < p\mu_d(1)$ for $p < p_c^{(2)}$ and $\mu_d(p) = p\mu_d(1)$ for $p > p_c^{(2)}$.

This is quite similar to what happens for directed percolation (see [22]) for which for $d \leq 2$ the number of open directed path is much smaller than its expected value for every p while when $d \geq 3$ a weak disorder phase exists.

Predicting anything about the critical dimension $d = 4$ is trickier, and all of this remains at a very speculative level. Whereas for directed percolation, the existence of a weak disorder phase can be proved with a two-line computation by computing the second moment of Z_N , it is a much more challenging question here.

Note that the relevant critical dimension for the problem we are interested in should be the one of the random-walk: $d = 4$ for the self-avoiding walk, and $d = 2$ and simple random walk in the oriented model, and not the one of the percolation process.

2. EXISTENCE OF THE QUENCHED CONNECTIVE CONSTANT AND MONOTONICITY PROPERTIES

2.1. Proof of Proposition 1.1. We prove in this Section that $\mu_d(p)$ is well defined. Some intermediate lemmata are proved for general infinite connected graphs.

Lemma 2.1. *For an infinite connected graph \mathcal{C} with bounded degree and $x \in \mathcal{C}$ (in the set of vertices), we define*

$$Z_N(x) := |\{ \text{self-avoiding path of length } N \text{ and starting from } x \text{ on } \mathcal{C} \}|.$$

Then the quantity

$$\limsup_{N \rightarrow \infty} (Z_N(x))^{1/N} \tag{2.1}$$

is a constant function of x , we call it $\mu(\mathcal{C})$.

Proof. As \mathcal{C} is by definition connected, is sufficient to show that $\limsup_{N \rightarrow \infty} (Z_N(x))^{1/N}$ takes the same value for every pair of neighbors. Let x and x' be connected by an edge of \mathcal{C} . Let $\bar{Z}_N(x)$ be the number of self-avoiding paths of length N starting from x and never

visiting x' . Let $Y_N(x, x')$ be the number of self-avoiding path of length N starting from x and ending at x' . One has trivially (noting that $Y_N(x, x') = Y_N(x', x)$)

$$\begin{aligned} Y_N(x, x') &\leq Z_N(x'), \\ \bar{Z}_N(x) &\leq Z_{N+1}(x'). \end{aligned} \quad (2.2)$$

The second inequality is simply saying that $\bar{Z}_N(x)$ also counts the number of paths of length $N + 1$ starting from x' whose first step is x .

Then decomposing $Z_N(x)$ along the possible options for the step k where it goes through x' one gets

$$Z_N(x) \leq \sum_{k=1}^N Y_k(x, x') Z_{N-k}(x') + \bar{Z}_N(x) \leq \sum_{k=1}^N Z_k(x') Z_{N-k}(x') + Z_{N+1}(x'). \quad (2.3)$$

which is enough to conclude that

$$\limsup_{N \rightarrow \infty} Z_N(x)^{1/N} \leq \limsup_{N \rightarrow \infty} Z_N(x')^{1/N}, \quad (2.4)$$

and thus by symmetry, that these two are equal. \square

What remains to be done is proving that when \mathcal{C} is a supercritical percolation cluster, $\mu(\mathcal{C})$ is a.s. non-random. A first step is to show that $\mu(\mathcal{C})$ is non-sensitive to individual edge-addition (and thus to edge removal).

Lemma 2.2. *Let \mathcal{C} be an infinite connected graph with bounded degree and $x, x' \in \mathcal{C}$ that are not linked by an edge. Then define*

- (i) \mathcal{C}' to be the graph constructed from \mathcal{C} by adding a new vertex called y and an edge (x, y) linking x to y .
- (ii) \mathcal{C}'' to be the graph with same set of vertices as \mathcal{C} , and an added edge: (x, x') .

We have

$$\mu(\mathcal{C}) = \mu(\mathcal{C}') = \mu(\mathcal{C}''). \quad (2.5)$$

Proof. We call Z'_N and Z''_N the number of self-avoiding path of length N on \mathcal{C}' resp. \mathcal{C}'' . Note that for $N \geq 2$, $Z_N(x) = Z'_N(x)$ and thus $\mu(\mathcal{C}) = \mu(\mathcal{C}')$.

Now consider the case of \mathcal{C}'' . Decomposing over paths that use the edge (x, x') and those who don't, one gets,

$$Z_N(x) \leq Z''_N(x) \leq Z_{N-1}(x') + Z_N(x). \quad (2.6)$$

Taking the above inequality to the power $\frac{1}{N}$ and passing to the lim sup one gets that

$$\limsup_{N \rightarrow \infty} (Z''_N(x))^{1/N} = \mu(\mathcal{C}), \quad (2.7)$$

which ends the proof. \square

We are now ready to conclude the proof of Proposition 1.1. In what follows \mathcal{C} denotes again the infinite connected component of the percolation process. By uniqueness of the infinite cluster, modifying the environment on any finite set of edge only add or delete finitely many edges to \mathcal{C} , so that the new cluster can be obtained from the old one by performing operation (i) or (ii) of Lemma 2.2 or their converse a finite number of time. Hence by Lemma 2.2, $\mu(\mathcal{C}(\omega))$ is measurable with respect to the tail-sigma algebra of the field $(\omega_e)_{e \in E_d}$, which is known to be trivial. Hence it is non-random. \square

2.2. Proof of monotonicity of $(\mu_d(p)/p\mu_d(1))$. To prove the monotonicity of the ration between quenched and annealed connectivity constant, we use a coupling argument that is quite standard. We couple the two measures \mathbb{P}_p and $\mathbb{P}_{p'}$ for $p_c < p < p' < 1$ by associating to edges e IID variables $X(e)$ (call \mathbb{E} the law of the field X) that are uniformly distributed in $[0, 1]$. Then one sets

$$\omega_p(e) = \mathbf{1}_{\{X(e) \leq p\}}, \quad \omega_{p'}(e) = \mathbf{1}_{\{X(e) \leq p'\}}. \quad (2.8)$$

With this construction, the infinite open clusters of ω_p and $\omega_{p'}$ satisfy $\mathcal{C}_p \subset \mathcal{C}_{p'}$. Moreover, if one sets

$$\mathcal{F}_{p'} = \sigma(\omega_{p'}(e), e \in \mathbb{Z}^d), \quad (2.9)$$

one has

$$\mathbb{E} [Z_N(\omega_p) \mid \mathcal{F}_{p'}] = \sum_{S \in \mathcal{S}_N} \mathbb{P} [\mathbf{1}_S \text{ is open for } \omega_p \mid \mathcal{F}_{p'}]. \quad (2.10)$$

The reader can then check that

$$\mathbb{P} [\mathbf{1}_S \text{ is open for } \omega_p \mid \mathcal{F}_{p'}] = \mathbf{1}_{S \text{ is open for } \omega_{p'}} (p/p')^N. \quad (2.11)$$

Summing over $S \in \mathcal{S}_N$ gives

$$\mathbb{E} [Z_N(\omega_p) \mid \mathcal{F}_{p'}] \leq (p/p')^N Z_N(\omega_{p'}). \quad (2.12)$$

Using the Borel-Cantelli Lemma, one gets that for all N large enough

$$Z_N(\omega_p) \leq N^2 (p/p')^N Z_N(\omega_{p'}), \quad (2.13)$$

which implies

$$\frac{\mu_d(p)}{p} \leq \frac{\mu_d(p')}{p'}. \quad (2.14)$$

□

3. PROOF OF THE MAIN RESULT: THEOREM 1.3

In this section we focus on the proof of the non-equality between quenched and annealed connectivity constant, or equation (1.11). For the proof of the strict monotonicity of $\mu_d(p)/p\mu_d$ we refer to Section (4).

3.1. About the proof. The main ingredients of the proof are fractional moment, coarse-graining and change of measure. The combination of these ingredients have have been used several time in a recent past in the study of disordered system in the aim of comparing quenched and annealed system. The method was first introduced in [13] for the study of disordered pinning on a hierarchical lattice. It was then improved in [24] (introduction of an efficient coarse-graining on a non-hierarchical setup) and in [14, 15] (improvement of the change of measure argument by introducing a multibody interaction). It has also been successfully adapted to a variety of model and we can cite of few contributions on random-walk pinning model [2, 3], directed-polymer in a random environment [21], stretched polymers [27], random-walk in a random environment [26]... (for technical details [21, Section 4] is probably the more related to what we are doing here).

The major difference between the all the above studied model and the one we are studying here is the amount of knowledge that one has on the annealed or pure model. For all the above cited model, the annealed version are either directed simple random walk or a mildly modified version of it (e.g. in [26, 27]) and the proof uses the acute knowledge that one has about simple random walk (e.g. the central limit theorem) to

draw conclusions. In [14, 15, 21, 26, 27], in the $(1 + 2)$ or 3 dimensional case, the need of a more refined change of measure is due to the fact that we are at the critical dimension, where extra-precision is needed.

On the contrary, here, even though we are not at the critical dimension (recall that we believe that the result also holds in dimension 3), similar refinements have to be used for a different reason. The problem is rendered more difficult by the fact that almost nothing has been rigorously proved for the planar self-avoiding walk in spite of numerous conjecture (e.g. we don't have a good control on $\mathbb{E}[Z_N]$ beyond the exponential scale, and we almost have no rigorous knowledge about the trajectory properties). For this reason, we need a method that covers all the worst-case scenarii. As a consequence we believe that the quantitative estimate that we derive from our method are almost irrelevant. The method does not rely on peculiar features of percolation or on the lattice and thus is quite easy to export to other 2-dimensional model (see Section 4).

The main novelties in the proof are:

- A new type of coarse-graining, that allows to treat the many possibilities of back-track for the walk (in [26, 27] even if the walk are allowed to backtrack the situation is different because they don't do it on large scale).
- A new type of change of measure, inspired from the one used in [21], but modified to adapt our new setup, and a new method to estimate the gain given with this change of measure.

The rest of the proof is organized as follows: In Section 3.2, we explain what we mean by fractional moment, and introduce our coarse-grained decomposition. It associates a coarse-grained lattice animal to each trajectory. This reduces the proof of (1.11) to Proposition 3.1 that controls the contribution of each animal. In Section 3.3 we give the main idea of the proof of Proposition 3.1, which is proved in Section 3.4 and 3.5 for small resp. large values of m .

3.2. Fractional moment method and Animal decomposition. Fractional moment is a technique extensively used by physicists that consists in estimating non-integer moment of a partition function in order to get non-trivial information about it. From now on we skip dependence in p in the notation when it does not affect understanding.

In our case, the fractional moment method consists in saying that to prove our result (1.11), it is sufficient to prove that there exists $\theta \in (0, 1)$ and $b < 1$ such that, for N large enough

$$\mathbb{E}[(Z_N)^\theta] \leq b^{N\theta} \mathbb{E}[Z_N]^\theta = [s_N(bp)^N]^\theta. \quad (3.1)$$

Indeed by Borel-Cantelli Lemma (combined with Markov inequality), (3.1) implies that a.s. for all N large enough

$$Z_N \leq N^{2/\theta} (bp)^N s_N, \quad (3.2)$$

so that passing to the limsup one gets

$$\mu_2(p) \leq bp\mu_2(1). \quad (3.3)$$

We consider the following coarse-graining procedure that to each path associate a lattice-animal on a rescaled lattice. Set

$$N_0 := \exp\left(\frac{C_2}{(1-p)^2}\right) \quad (3.4)$$

where C_2 is a constant (independent of p , its value will be fixed at the end of the proof), and let us partition the set of edges E_d into squares of side length N_0 . More precisely, let $r(e)$ denote the smaller end (for the lexicographical order on \mathbb{Z}^2) of an edge $e \in E_d$, and for $x \in \mathbb{Z}^2$ one defines

$$I_x := \{e \in E_d \mid r(e) \in (x + [0, N_0)^2)\}. \quad (3.5)$$

Now, one associate to a path S the set of squares I_x that it visits as follows Set

$$A(S) := \{x \in \mathbb{Z}^d \mid \exists n \in [0, N-1], (S_n, S_{n+1}) \in I_x\}. \quad (3.6)$$

Note that $A(S)$ is a connected subset of \mathbb{Z}^2 that contains the origin (sometimes called a site-animal) and that

$$\lceil N/N_0^2 \rceil \leq |A(S)| \leq 9\lceil N/N_0 \rceil. \quad (3.7)$$

The upper-bound comes from the fact that in N_0 steps, one cannot visit more than 9 different I_x 's (the one from which one starts and all the neighbors), whereas the lower-bound simply uses the fact that there are only N_0^2 sites to visit in each square. From now on, one drops the integer parts from the notation for simplicity. Set \mathfrak{A}_m to be the set of connected subset of \mathbb{Z}^2 of size m containing the origin and $a_m := |\mathfrak{A}_m|$. For each animal \mathcal{A} one sets

$$\mathcal{S}_N(\mathcal{A}) := \{S \in \mathcal{S}_N \mid A(S) = \mathcal{A}\}. \quad (3.8)$$

Then one decompose the partition function according to the contribution of each animal,

$$Z_N = \sum_{m=N/N_0^2}^{9((N/N_0)+1)} \sum_{\mathcal{A} \in \mathfrak{A}_m} \sum_{S \in \mathcal{S}_N(\mathcal{A})} \mathbf{1}\{S \text{ is open}\} =: \sum_{m=N/N_0^2}^{9((N/N_0)+1)} \sum_{\mathcal{A} \in \mathfrak{A}_m} Z_N(\mathcal{A}). \quad (3.9)$$

We use the following trick: for any $\theta < 1$ and any sequence of positive number $(a_n)_{n \in \mathbb{N}}$ one has

$$\left(\sum_{n \in \mathbb{N}} a_n\right)^\theta \leq \sum_{n \in \mathbb{N}} a_n^\theta. \quad (3.10)$$

Thus applying this to (3.9) and averaging one gets:

$$\mathbb{E} \left[Z_N^\theta \right] \leq \sum_{m=N/N_0^2}^{9((N/N_0)+1)} \sum_{\mathcal{A} \in \mathfrak{A}_m} \mathbb{E} \left[Z_N(\mathcal{A})^\theta \right]. \quad (3.11)$$

There are at most exponentially many animals of size m . Here, we use the crude estimate $a_m \leq 49^m$ (see e.g. [16, (2.4) pp 81], the definition of lattice animal given there differs but the bound still applies). Hence

$$\mathbb{E} \left[Z_N^\theta \right] \leq N \max_{m \in [N/N_0^2, 9((N/N_0)+1)]} 49^m \max_{\mathcal{A} \in \mathfrak{A}_m} \mathbb{E} \left[Z_N(\mathcal{A})^\theta \right]. \quad (3.12)$$

Thus, in order to prove (3.1) it is sufficient to prove that e.g. $\mathbb{E} \left[Z_N(\mathcal{A})^\theta \right] \leq 100^{-m} (ps_n)^\theta$ for every m and \mathcal{A} . This is the key part of the proof.

Proposition 3.1. *For $\theta = 1/2$, for every $\mathcal{A} \in \mathfrak{A}_m$*

$$\mathbb{E} \left[Z_N(\mathcal{A})^\theta \right] \leq p^{N\theta} s_N^\theta 100^{-m}, \quad (3.13)$$

if the constant C_2 is chosen large enough.

The above proposition combined with equation (3.12) implies that

$$\mathbb{E} \left[Z_N^\theta \right] \leq N p^{N\theta} s_N^\theta 2^{-N/N_0^2}, \quad (3.14)$$

which implies (3.1) for $b = 4^{-N/N_0^2}$ and N large enough. Thus from (3.1)-(3.3) we get

$$\mu_2(p) = \limsup_{N \rightarrow \infty} (Z_N)^{1/N} \leq 4^{-1/N_0^2} p \mu_2(1) < p \mu_2(1). \quad (3.15)$$

Hence there exists a constant c such that for all p

$$p \mu_2(1) - \mu_2(p) \leq \frac{c}{N_0^2} \geq c \exp \left(-\frac{2C_2}{(1-p)^2} \right). \quad (3.16)$$

3.3. Change of measure strategies. Let us explain our strategy behind the proof of Proposition 3.1. It is based on a change of measure argument. The fundamental idea is that if $\mathbb{E}[\sqrt{Z_N(\mathcal{A})}]$ is much smaller than $\sqrt{\mathbb{E}[Z_N(\mathcal{A})]}$, it must be because there is a small set of ω (of small \mathbb{P} probability), that gives a the main contribution to $\mathbb{E}[Z_N(\mathcal{A})]$. We want to introduce a random variable $f_{\mathcal{A}}(\omega)$ which takes low value for these untypical environment and use the Cauchy-Schwartz inequality as follows.

Lemma 3.2. *For any \mathcal{A} and any positive random variable $f_{\mathcal{A}}$ one has*

$$\mathbb{E} \left[Z_N(\mathcal{A})^{1/2} \right] \leq \mathbb{E} [f_{\mathcal{A}} Z_N(\mathcal{A})]^{1/2} (\mathbb{E} [(f_{\mathcal{A}})^{-1}])^{1/2}. \quad (3.17)$$

Proof. It is just Cauchy-Schwartz inequality applied to the product $(Z_N(\mathcal{A}) f_{\mathcal{A}})^{1/2} \times f_{\mathcal{A}}^{-1/2}$. \square

Note that if $f_{\mathcal{A}}$ has finite expectation, it can be thought as a probability density after renormalization so that this operation can indeed be interpreted as a change of measure. The main problem then is to find an efficient change of measure for which the cost of the change $\mathbb{E} [(f_{\mathcal{A}})^{-1}]$ is much less than the benefit one gets on $\mathbb{E} [f_{\mathcal{A}} Z_N(\mathcal{A})]$.

In order to get an exponential decay in m for $\mathbb{E} [Z_N(\mathcal{A})^{1/2}] / \mathbb{E} [Z_N(\mathcal{A})]^{1/2}$, the good choice is to choose $f_{\mathcal{A}}$ as a product of functions of the environment of each block $(\omega_e)_{e \in I_x}$, over all $x \in \mathcal{A}$.

A possibility is to diminish the intensity of open edges in $\cup_{x \in \mathcal{A}} I_x$, simply by choosing

$$f_{\mathcal{A}}(\omega) \asymp \lambda^{\# \text{ open edges}}, \quad (3.18)$$

for some $\lambda < 1$. This turns out to be a good choice when the animal \mathcal{A} considered is relatively small, but it does not give a good result when $m = |\mathcal{A}|$ is of order N/N_0 , even after optimizing the value of λ . A more efficient strategy in that case is to induce negative correlation that decay with the distance between opening of different edges instead of reducing the intensity of edge opening.

This idea was first used in [14]. There and in all related works, the induced negative correlation was chosen to be proportional to the Green function of the underlying process (either a renewal process in [14] or a directed random-walk in [21, 26, 27]). Here the situation is a bit different as one does not have any information on the underlying process, and therefore the choice of correlation (i.e. of the coefficients in the quadratic form Q in equation (3.29)) is done via an optimization procedure so that it lowers significantly the probability of $\mathbb{P}[S \text{ is open}]$ for every possible path (and not only the more probable ones).

We adopt the first strategy when $m \leq N/[N_0(\log N_0)^{1/4}]$ and the second one when $m > [N_0(\log N_0)^{1/4}]$.

3.4. Proof of Proposition 3.1 for small values of m . In this section we assume that

$$m \leq N/[N_0(\log N_0)^{1/4}]. \quad (3.19)$$

We choose to modify the environment in

$$I_{\mathcal{A}} := \bigcup_{x \in \mathcal{A}} I_x,$$

by augmenting the intensity of the edge dilution. We choose the probability of an edge being open under the new measure to be equal to

$$p' := \frac{\lambda p}{1 - p(1 - \lambda)}$$

where $\lambda < 1$ is chosen such that

$$(1 - \lambda)\sqrt{1 - p}N_0 = 1. \quad (3.20)$$

As there are $2N_0^2$ edges in each block I_x , the density function corresponding to this change of measure is given by

$$f_{\mathcal{A}} := \frac{\lambda^{\#\{\text{open edges in } I_{\mathcal{A}}\}}}{[1 - p(1 - \lambda)]^{2mN_0^2}}. \quad (3.21)$$

Then we can estimate the cost of the change of measure

$$\begin{aligned} \mathbb{E}[f_{\mathcal{A}}^{-1}] &= \left[\left(1 + \frac{p}{\lambda}(1 - \lambda)\right) (1 - p(1 - \lambda)) \right]^{2mN_0^2} \\ &= \left(1 + \frac{p}{\lambda}(1 - p)(1 - \lambda)^2\right)^{2mN_0^2} \leq \exp\left(\frac{2p}{\lambda}m\right) \leq 10^m, \end{aligned} \quad (3.22)$$

where for the first inequality, we used $\log(1 + x) \leq x$ and the identity (3.20), and for the second one, the fact that λ is close to one (if C_2 is chosen large enough).

The probability for a path of length N whose edges are all in $I_{\mathcal{A}}$ to be open under the modified measure is equal to $[\lambda p/(1 - p(1 - \lambda))]^N$ so that

$$\mathbb{E}[f_{\mathcal{A}} Z_N(\mathcal{A})] = |\mathcal{S}_N(\mathcal{A})| p^N \left(\frac{\lambda}{1 - p(1 - \lambda)} \right)^N. \quad (3.23)$$

Then we remark that $|\mathcal{S}_N(\mathcal{A})| \leq s_N$ trivially and that

$$\begin{aligned} \left(\frac{\lambda}{1 - p(1 - \lambda)} \right)^N &= \left(1 - \frac{(1 - \lambda)(1 - p)}{1 - p(1 - \lambda)} \right)^N \leq \exp(-N(1 - \lambda)(1 - p)) \\ &\stackrel{(3.19)}{\leq} \exp\left(-mN_0(\log N_0)^{1/4}(1 - \lambda)(1 - p)\right) \\ &\stackrel{(3.20)}{=} \exp\left(-m(\log N_0)^{1/4}\sqrt{1 - p}\right) \stackrel{(3.4)}{=} \exp(-mC_2^{1/2}). \end{aligned} \quad (3.24)$$

Hence if C_2 is large enough

$$\mathbb{E}[f_{\mathcal{A}} Z_N(\mathcal{A})] \leq s_N p^N \exp(-mC_2^{1/2}) \leq s_N p^N 10^{-5m}. \quad (3.25)$$

Combining this with (3.22) and Lemma 3.2, we get (3.13). \square

Remark 3.3. Let us now comment on why we believe that (3.16) is suboptimal. The idea is that in fact, if the typical scaling of a self-avoiding path is N^α , then (at heuristic level) the typical number of blocs visited m is of order $N/(N_0^{1/\alpha})$, and the above argument works for a much lower value of N_0 (of order $(1-p)^{\frac{2\alpha}{2(1-\alpha)}}$), giving then a much better upper bound for all $\mu_2(p)$. Bringing this kind of argument on a rigorous ground would require very detailed knowledge of the behavior of the self-avoiding walk.

The same heuristic argument also indicates $\mu_3(p) < p\mu_3(1)$ provided the walk scales like N^α with $\alpha < 2/3$ (which is what has been predicted so far in the physics literature).

3.5. Proof of Proposition 3.1 for large values of m . Even when trying to optimize over the value of λ or when taking a much larger value for N_0 the preceding method fails when the size of animals is of order N/N_0 , and we have to apply a different method in this case. Throughout this section we will consider that

$$m > N/[N_0(\log N_0)^{1/4}]. \quad (3.26)$$

Our proof is still based on Lemma 3.2, the construction of our $f_{\mathcal{A}}$ is a bit more complicated in that case, and for notational convenience we do not normalize it.

First, given an animal \mathcal{A} of size m one can extract a set of vertices $\bar{\mathcal{A}}$ of size $m/9$ such that the vertices of \mathcal{A} have disjoint l_∞ neighborhood, i.e. such that

$$\forall x, y \in \bar{\mathcal{A}}, \quad x \neq y, \quad |x - y|_\infty \geq 3, \quad (3.27)$$

where $|x|_\infty = \max(|x_1|, |x_2|)$.

For instance one can construct $\bar{\mathcal{A}}$ by picking vertices in \mathcal{A} iteratively as follows: at each step we pick the smallest available vertex according to the lexicographical order in \mathbb{Z}^d , and make all the vertices at distance 2 or less of this vertex unavailable for future picks. As at most 9 vertices are made unavailable at each step, we can keep this procedure going during $m/9$ steps and get $\bar{\mathcal{A}}$. For $x \in \mathbb{Z}^2$ set

$$\bar{I}_x := I_x \cup \left(\bigcup_{\{y \mid |y-x|_\infty=1\}} I_x \right). \quad (3.28)$$

We define the distance d between edges to be the Euclidean distance between their mid-points. Given K a (large) constant, we define f_x to be a function of ω , that depends only on $\omega|_{\bar{I}_x}$: first set Q_x to be the following random quadratic form

$$Q_x(\omega) := \frac{1}{(1-p)N_0\sqrt{\log N_0}} \sum_{\substack{e, e' \in \bar{I}_x \\ e' \neq e}} \frac{1}{d(e, e')} (\omega(e) - p)(\omega(e') - p), \quad (3.29)$$

and then define

$$f_x(\omega) := \exp(-K \mathbf{1}_{\{Q_x(\omega) \geq e^{K^2}\}}). \quad (3.30)$$

Finally, set

$$f_{\mathcal{A}}(\omega) := \prod_{x \in \bar{\mathcal{A}}} f_x(\omega). \quad (3.31)$$

Then in order to use Lemma 3.2, one needs to bound from above $\mathbb{E}[(f_{\mathcal{A}})^{-1}]$.

Lemma 3.4. *If K is chosen sufficiently large, for every $A \in \mathfrak{A}_m$*

$$\mathbb{E}[(f_A)^{-1}] \leq 2^{m/9}. \quad (3.32)$$

Proof. The function $f_A(\omega)$ is a product of $m/9$ IID random variables $(f_x(\omega), x \in \bar{A})$, (this is due to the fact that due to our choice for \bar{A} , the blocks $(\bar{I}_x)_{x \in \bar{A}}$ are disjoint). Thus

$$\mathbb{E}[(f_A)^{-1}] = \mathbb{E}[(f_0)^{-1}]^{m/9}. \quad (3.33)$$

It remains to prove that $\mathbb{E}[(f_0)^{-1}] \leq 2$ and for this purpose, it is sufficient to estimate the variance of $Q_x(\omega)$. First note that $\mathbb{E}[Q_0(\omega)] = 0$ and that only the diagonal terms of the double sum that appears when developing $Q_0^2(\omega) = 0$ contributes to the second moment. Note also that the maximal distance between two edges in \bar{I}_0 is less than $5N_0$ so that

$$\begin{aligned} \mathbb{E}[Q_0(\omega)^2] &= \frac{1}{(1-p)^2 N_0^2 \log N_0} \sum_{\substack{e, e' \in \bar{I}_0 \\ e' \neq e}} \frac{1}{d(e, e')^2} p^2 (1-p)^2 \\ &\leq \frac{1}{N_0^2 \log N_0} \sum_{e \in \bar{I}_x} \sum_{\{e' \neq e \mid d(e, e') \leq 5N_0\}} \frac{1}{d(e, e')^2} \leq C_1. \end{aligned} \quad (3.34)$$

where C_1 is a universal constant that is independent of p and N_0 . Thus, by Chebycheff inequality, if K is large enough (independently of all parameter of the problem)

$$\mathbb{E}[(f_0(\omega))^{-1}] = 1 + (e^K - 1) \mathbb{P}[Q_0(\omega) \geq e^{K^2}] \leq 1 + C_1(e^K - 1)e^{-2K^2} \leq 2, \quad (3.35)$$

provided that K has been chosen large enough. \square

We are left with estimating $\mathbb{E}[f_A Z_N(\mathcal{A})]$, which is the more delicate part. We do so by bounding uniformly the contribution of each path.

Lemma 3.5. *For any $S \in \mathcal{S}_N(\mathcal{A})$*

$$\mathbb{E}[f_A(\omega) \mathbf{1}_{\{S \text{ is open}\}}] = p^N \mathbb{E}[f_A(\omega) \mid S \text{ is open}] \leq p^N 20000^{-m}. \quad (3.36)$$

The lemma, combined with the trivial bound $|\mathcal{S}_N(\mathcal{A})| \leq s_N$ gives,

$$\mathbb{E}[f_A(\omega) Z_N(\mathcal{A})] \leq p^N s_N 20000^{-m}, \quad (3.37)$$

so that together with Lemmata 3.2 and 3.4, one obtains

$$\mathbb{E}[Z_N(\mathcal{A})^{-1/2}] \leq p^{N/2} s_N^{N/2} 100^{-m}, \quad (3.38)$$

which proves Proposition 3.1.

Proof of Lemma 3.5. Note that even after conditioning on S being open, $f_A(\omega)$ is still a product of independent variables (though the $f_x(\omega)$ are not identically distributed any more), so that

$$\mathbb{E}[f_A(\omega) \mid S \text{ is open}] = \prod_{x \in \bar{A}} \mathbb{E}[f_x(\omega) \mid S \text{ is open}]. \quad (3.39)$$

Our idea is then to show that most of the term in the product $\mathbb{E}[f_x(\omega) \mid S \text{ is open}]$ are small. We do so by showing that conditioning to the event $\{S \text{ is open}\}$ makes the expectation of Q_x large whereas its variance stays relatively small. The problem is that both expectation and variance of $Q_x(\omega)$ may grow when additional edges are conditioned

on being open, and things becomes difficult to control when the number of edge that S visits in the block \bar{I}_x is much larger than N_0 . This is the reason why we restrict the use of this method to large values of m : we show that $\mathbb{E}[f_x(\omega) \mid S \text{ is open}]$ is small only for blocks for which the number of edges visited by S is not too large.

Set

$$\bar{A}(S) := \{x \in \bar{\mathcal{A}} \mid |S \cap \bar{I}_x| \leq 20N_0(\log N_0)^{1/4}\}, \quad (3.40)$$

where here S is considered as a set of edges. As the total number of edge in S is $N \leq mN_0(\log N_0)^{1/4}$ and the \bar{I}_x are disjoint, one has

$$|\bar{\mathcal{A}} \setminus \bar{A}(S)| \leq \frac{N}{20N_0(\log N_0)^{1/4}} \leq m/20. \quad (3.41)$$

and hence $|\bar{A}(S)| \geq m/20$.

Thus from (3.39) and using the fact that all the terms in the product are small than one, to prove (3.36) it is enough to prove that for each $x \in \bar{A}(S)$

$$\mathbb{E}[f_x(\omega) \mid S \text{ is open}] \leq 20000^{-20}. \quad (3.42)$$

Assume in the rest of the proof that $x \in \bar{A}(S)$. The definition of f_x gives

$$\mathbb{E}[f_x(\omega) \mid S \text{ is open}] = e^{-K} + (1 - e^{-K})\mathbb{P}[Q_x(\omega) < e^{K^2} \mid S \text{ is open}], \quad (3.43)$$

and thus if K is chosen large enough, it is sufficient to prove that

$$\mathbb{P}[Q_x(\omega) < e^{K^2} \mid S \text{ is open}] \leq 20000^{-21}. \quad (3.44)$$

To obtain such an estimate it is sufficient to compute the two first moments of $Q_x(\omega)$ under the conditioned measure. To keep notation light, we write \mathbb{P}_S for $\mathbb{P}[\cdot \mid S \text{ is open}]$.

To show that the first moment is large, one needs to extract a long path of adjacent edges. Set $S^{(x)}$ to be a path a length N_0 defined as follows: if $x = 0$ then $(S_n^{(0)})_{n \in [0, N_0]} := (S_n)_{n \in [0, N_0]}$; for all other values of x ,

- one sets τ_x be the first time that S hits I_x (note that $\tau_x \geq N_0$).
- one defines $(S_n^{(x)})_{n \in [0, N_0]} := (S_{n+\tau_x-N_0})_{n \in [0, N_0]}$.

Note that $S^{(x)}$ has all its edges in \bar{I}_x .

Under the measure \mathbb{P}_S , the $\omega(e)$ are independents, a.s. equal to one if $e \in S$, and distributed as Bernoulli variables of parameter p otherwise. Thus

$$\mathbb{E}_S[Q_x(\omega)] = \frac{(1-p)}{N_0\sqrt{\log N_0}} \sum_{\substack{e, e' \in S \\ e \neq e'}} \frac{1}{d(e, e')} \geq \frac{(1-p)}{N_0\sqrt{\log N_0}} \sum_{\substack{e, e' \in S^{(x)} \\ e \neq e'}} \frac{1}{d(e, e')}. \quad (3.45)$$

Now for every edge $e \in S^{(x)}$, as the trajectory $S^{(x)}$ can't be more than ballistic

$$\sum_{e' \in S^{(x)} \setminus \{e\}} \frac{1}{d(e, e')} \geq \sum_{n=1}^{N_0-1} \frac{1}{n} \geq \log N_0. \quad (3.46)$$

Therefore

$$\mathbb{E}_S[Q_x(\omega)] \geq (1-p)\sqrt{\log N_0}. \quad (3.47)$$

Let us now bound from above the variance $\text{Var}_{\mathbb{P}_S}(Q_x(\omega))$. Most of the terms in the resulting sum appear in the non-conditioned case, we have to check that the additional terms generated by the conditioning only give a small contribution.

$$\begin{aligned} \text{Var}_{\mathbb{P}_S}[Q_x(\omega)] &= \frac{1}{(1-p)^2 N_0^2 \log N_0} \sum_{\substack{e, e' \in \bar{I}_x \setminus S \\ e' \neq e}} \frac{1}{d(e, e')^2} p^2 (1-p)^2 \\ &\quad + \frac{1}{(1-p)^2 N_0^2 \log N_0} \sum_{e \in \bar{I}_x \setminus S} p(1-p)^3 \left(\sum_{e' \in S \cap \bar{I}_x} \frac{1}{d(e', e)} \right)^2. \end{aligned} \quad (3.48)$$

The first term in the r.h.s. is less than $\text{Var}_{\mathbb{P}}(Q_x)$ (see equation (3.34), it is the same sum with some missing terms) and thus is bounded above by C_1 . Using Cauchy-Schwartz inequality we can bound the second term from above as follows

$$\begin{aligned} \sum_{e \in \bar{I}_x \setminus S} \left(\sum_{e' \in S \cap \bar{I}_x} \frac{1}{d(e', e)} \right)^2 &\leq |\bar{I}_x \cap S| \sum_{e \in \bar{I}_x \setminus S^{(x)}} \sum_{e' \in S \cap \bar{I}_x} \frac{1}{d(e', e)^2} \\ &\leq |\bar{I}_x \cap S| \sum_{e' \in |\bar{I}_x \setminus S^{(x)}| \setminus \{e \neq e' \mid d(e, e') \leq 5N_0\}} \frac{1}{d(e', e)^2} \leq C_1 |\bar{I}_x \cap S|^2 \log N_0. \end{aligned} \quad (3.49)$$

And thus as for $x \in \bar{\mathcal{A}}(S)$, $|\bar{I}_x \cap S| \leq 20N_0(\log N_0)^{1/4}$,

$$\text{Var}_{\mathbb{P}_S}(Q_x(\omega)) \leq C_1(1 + 400(1-p)(\log N_0)^{1/2}) \quad (3.50)$$

Recall that $N_0 := \exp(\frac{C_2}{(1-p)^2})$, and set the constant C_2 to be such that

$$\mathbb{E}_S[Q_x(\omega)] \geq (1-p)\sqrt{\log N_0} = \sqrt{C_2} = 2e^{K^2}. \quad (3.51)$$

Then one has, by Chebycheff inequality and (3.50)

$$\begin{aligned} \mathbb{P}_S(Q_x(\omega) \leq e^{K^2}) &\leq \mathbb{P}_S(|Q_x(\omega) - \mathbb{E}_S[Q_x(\omega)]| \geq e^{K^2}) \\ &\leq \text{Var}_{\mathbb{P}_S}(Q_x(\omega)) e^{-2K^2} \leq 401C_1 e^{-K^2}, \end{aligned} \quad (3.52)$$

which proves (3.44) if K is large enough, and ends the proof. \square

4. SOME OTHER MODELS TO WHICH THE PROOF CAN ADAPT

Note that our proof, though technical, did not use many specifics of the model. The key point in the proof is that the r.h.s. of (3.45) diverges with N_0 and this is where the crucial fact that the lattice is 2-dimensional is used. For this reason, our result extends readily to any kind of two-dimensional lattice (e.g. triangular, honeycomb, lattices with spread-out connections). Moreover the proof would also work with only minor modification for a large variety of 2-dimensional model. Without trying to describe a meta-model that would include all of these, we give here some examples that could be of interest.

4.1. Starting from a supercritical percolation cluster: proof that

$p \mapsto \mu_2(p)/(p\mu_2(1))$ is (strictly) increasing. Consider $p < p'$, both in the interval $(p_c, 1)$. We couple to percolation environment ω_p and $\omega_{p'}$ as we did in Section 2.2. Applying exactly the same proof as above but replacing \mathbb{Z}^2 by the infinite percolation cluster of $\omega_{p'}$ and s_n by $Z_N(\omega_{p'})$ gives us that there exists $b < 1$ such that

$$\mathbb{E}[(Z_N(\omega_p))^{1/2} \mid \mathcal{F}_{p'}] = (b^N (p'/p)^N Z_N(\omega_{p'}))^{1/2}. \quad (4.1)$$

From this and the Borel-Cantelli Lemma we get that almost surely

$$\limsup_{N \rightarrow \infty} (Z_N(\omega_p))^{1/N} < (p'/p)^N \limsup_{N \rightarrow \infty} (Z_N(\omega_{p'}))^{1/N}. \quad (4.2)$$

This implies

$$\frac{\mu_2(p)}{p} < \frac{\mu_2(p')}{p'}. \quad (4.3)$$

4.2. Site Percolation. One consider the equivalent of the model studied in the core of this paper, but with the disorder $(\omega(x))_{x \in \mathbb{Z}^2}$ lying on the sites of \mathbb{Z}^2 rather than on the edges. One say that a self-avoiding path is open if all the sites visited by the path are open. One can check readily that using exactly the same arguments, one can prove the existence of the quenched connective constant (with \limsup) and the fact that it differs from the annealed one for every p . One can furthermore adapt Section 4.1 to show that the ratio of the two connective constants is an increasing function of p .

4.3. Self-avoiding walk in a Random Potential. This is a generalization of the preceding model. Let $\omega(x)$ a collection of IID random variables of zero mean and unit variance, indexed by sites of \mathbb{Z}^2 , \mathbb{P} denote the joint law, which satisfies

$$e^{\lambda(\beta)} := \mathbb{E} \left[e^{\beta \omega(0)} \right] < \infty$$

for all $\beta \in \mathbb{R}$.

One is interested in the probability measure on \mathcal{S}_N where each path S has a probability proportional to

$$\Pi_\omega(S) = \exp \left(\beta \sum_{n=0}^N \omega(S_n) \right). \quad (4.4)$$

Physically, $(-\omega)$ corresponds to an energy attached to each site, and β is the inverse temperature. We are interested in the growth rate of the partition function of this model:

$$Z_N = \sum_{S \in \mathcal{S}_N} \Pi_\omega(S). \quad (4.5)$$

Theorem 4.1. *For any $\beta > 0$ there exist a positive constant $c(\beta)$ such that*

$$\limsup_{N \rightarrow \infty} (Z_N)^{1/N} < \limsup_{N \rightarrow \infty} \mathbb{E} [Z_N]^{1/N} = \mathbb{E} \left[e^{\beta \omega(0)} \right]. \quad (4.6)$$

Let us mention a few guidelines to adapt the proof:

- One must choose $N_0 := \exp(-C_2/\beta^4)$.
- In section 3.4, instead of augmenting dilution, one tilts the random variable in $I_{\mathcal{A}}$, choosing

$$f_{\mathcal{A}}(\omega) := \exp \left(- \left(\sum_{x \in I_{\mathcal{A}}} \delta \omega_x \right) - N_0^2 \lambda(-\delta) \right), \quad (4.7)$$

where the definition of $I_{\mathcal{A}}$ and I_x has been adapted so that they are sets of points rather than sets of edge. The right value to choose for δ is $\delta = (1/N_0)$.

- In section 3.5, one considers

$$Q_x(\omega) := \frac{1}{N_0 \sqrt{\log N_0}} \sum_{\substack{z, z' \in \bar{I}_x \\ z' \neq z}} \frac{\omega_z \omega_{z'}}{|z - z'|}. \quad (4.8)$$

Having this at hand, the computation are essentially the same that in Section 3.

4.4. Lattice trees/Lattice animals on a dilute network. A lattice tree of size N on \mathbb{Z}^2 is a finite connected subgraph of \mathbb{Z}^2 with N vertices and no cycles. We denote by \mathcal{T}_N the number of (unlabeled) lattice trees of size N containing the origin in \mathbb{Z}^2 and $t_n = |\mathcal{T}_n|$. It is known (see e.g. [19] where lattice trees are studied as a model for branching polymers) that there exists a constant μ such that

$$\lim_{N \rightarrow \infty} (t_N)^{\frac{1}{N}} = \mu. \quad (4.9)$$

The method presented above also allows to give an upper-bound on the number of lattice trees present on an infinite percolation. We say that a lattice tree is open if all the edges that constitute it are open. Given $(\omega(e))_{e \in E_d}$ a realization of the edge dilution process, one defines

$$Z_N := \sum_{T \in \mathcal{T}_N} \mathbf{1}_{\{T \text{ is open}\}}, \quad (4.10)$$

and $Z_N(x)$ to be the same thing but for lattice trees that contains x . It can be shown, as we have done for self-avoiding path, that the upper-growth rate

$$\limsup_{N \rightarrow \infty} (Z_N(x))^{1/N} \quad (4.11)$$

is constant on the infinite percolation cluster and non-random. Using exactly the same proof as in Section 3, one can further prove,

Theorem 4.2. *For any $p \in (1/2, 1)$ one has \mathbb{P}_p a.s. for all point $x \in C$,*

$$\limsup_{N \rightarrow \infty} (Z_N(x))^{1/N} < \limsup_{N \rightarrow \infty} (\mathbb{E}_p [Z_N(x)])^{1/N} = p\mu. \quad (4.12)$$

Proof. Maybe the only point that needs to be explained in the adaptation of the proof is how one chooses $S^{(x)}$ appearing in (3.45). For $x = 0$ we choose arbitrarily a paths of length N_0 moving away from the root (there has to be at least one if $N \geq N_0^2$). For the other values of x we fix $S_0^{(x)}$ to be a point of the tree that lies in I_x , and $S^{(x)}$ to be first N_0 steps on the paths from this point towards the root, i.e. the origin. \square

A similar result could be stated for lattice-animals on the supercritical percolation cluster.

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